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Geometrical analysis of free rotation of a rigid body

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Abstract. Rotational motion of a free rigid body without action of external torque is governed by the Euler equation, which is reformulated as a geodesic equation by introducing a Riemannian metric and connection on the Lie group $SO(3)$. The aim of this work is to investigate the stability of the motion from the viewpoint of geodesic variation and also to study the relationship between the instability and the Riemannian sectional curvatures of variable signs.

The stability theorem of steady rotations known in the mechanics (*dynamical* property) is recovered by solving the Jacobi equation for the variational field $\boldsymbol{Y}$ (a *geometrical* field). Existence of a conjugate point for the rigid body of any shape is confirmed, and the condition for any Jacobi field concerning the steady rotations such that $\boldsymbol{Y}|_{t=0} = 0$ to have a conjugate time is derived. The sectional curvatures K's are calculated. For the stable steady rotation, the curvatures K's take either positive definite values, or both positive and negative values in oscillatory manner, depending on the inertia tensor. However, the time averages $\bar{K}$'s are always positive for any $\boldsymbol{Y}$ in the linearly stable case, while there exist $\boldsymbol{Y}$'s which make $\bar{K}$ negative in the case of linear instability.

1. Introduction

Hamiltonian formulation based on the Lie group theory is extensively developed and has recently been applied to various physical systems [1, 2]. They are called the *Lie–Poisson* systems and can be reformulated as geodesic equations on the corresponding Lie group manifolds if the Hamiltonian function is quadratic [3]. This reformulation has a merit that the mathematical theory of geodesic instability can be applied. It provides a novel analysis of dynamical systems in terms of the Riemannian geometry and the theory of Lie group.

Rotational motion of a free rigid body (without action of external force) is governed by the Euler equation,

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3 \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1 \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2 \end{aligned} \tag{1}$$

where $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the angular velocity in the *body* coordinate system and I_i's ($i = 1, 2, 3$) are the principal values of the inertia tensor. Equation (1) is a geodesic equation in the sense that it is the Euler–Lagrange equation for the variational principle $\delta\int T\,\mathrm{d}t = 0$, which is equivalent to $\delta\int\sqrt{T}\,\mathrm{d}t = 0$ apart from reparametrization, where T

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is kinetic energy of the rigid body. Since the solution $\boldsymbol{\Omega}(t)$ is explicitly represented in terms of the Jacobian elliptic functions [4, 7], equation (1) is completely integrable. The Euler equation is a typical Lie–Poisson system and provides the basis of general formulation. However, despite its importance, an instability analysis based on the Riemannian geometry has not been made to date (to the authors' knowledge). In this paper, we investigate the Riemannian structure on the rotation group $SO(3)$ and instability of the rigid-body motion by formulating equation (1) as a geodesic equation.

2. Riemannian geometry on $SO(3)$

Before introducing the Riemannian geometry on the group manifold, let us summarize the Lie–Poisson formulation of equation (1). We denote by $\boldsymbol{P} \in \mathbb{R}^3$ the *material* points of the rigid body with the coordinates (P_1, P_2, P_3) relative to the frame fixed to the body (*body system*). Corresponding spatial points relative to the coordinate system fixed in the space (*spatial system*) are denoted by $\boldsymbol{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$. Both systems are assumed to have a common fixed origin. By the definition of the rigid body, there exists an orbit $M(t)(t \in \mathbb{R}^1)$ on $SO(3)$ such that $\boldsymbol{p} = M(t)\boldsymbol{P}$. It is known in the classical analysis that the motion of the angular velocity vector is time-periodic in the body system, while the body's motion is quasiperiodic relative to the fixed space [2, 4]. By defining the angular velocity $\boldsymbol{\Omega} \in \mathbb{R}^3$ and the angular momentum $\boldsymbol{\Pi} \in \mathbb{R}^3$ relative to the body system, it can be shown in the representation on $\mathbb{R}^3$ that $\boldsymbol{\Omega}$ is an element of the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$, while $\boldsymbol{\Pi}$ belongs to its dual space $\mathfrak{so}(3)^*$ [2]. The duality between $\boldsymbol{\Omega}$ and $\boldsymbol{\Pi}$ is confirmed by one-to-one linear correspondence $\boldsymbol{\Pi} = \mathbf{I}(\boldsymbol{\Omega})$ where $\mathbf{I} := \mathrm{diag}(I_1, I_2, I_3)$ is the inertia tensor [2, 5]. Following the standard method, if we take the Hamiltonian given by $H(\boldsymbol{\Pi}) = \frac{1}{2}\{\boldsymbol{\Pi} \cdot \mathbf{I}^{-1}(\boldsymbol{\Pi})\} = \frac{1}{2}\sum_{j=1}^{3} I_j{}^{-1}\Pi_j^2 \in C^\infty(\mathfrak{so}(3)^*)$, it is readily verified that the governing equation for $\boldsymbol{\Pi}$ is given as the Hamilton's equation $\frac{\mathrm{d}}{\mathrm{d}t}F = \{F, H\}$ where $\{*, *\}$ is the *Lie–Poisson bracket* [2].

Because the above Hamiltonian is quadratic, following the theorem by Arnold [3], an inner product at the identity element $e \in SO(3)$ is naturally introduced as $\langle\langle \boldsymbol{\Omega}, \boldsymbol{\Omega}' \rangle\rangle_e = \sum_{j=1}^{3} I_j \Omega_j \Omega_j'$ where $\boldsymbol{\Omega}, \boldsymbol{\Omega}' \in \mathfrak{so}(3)$. Consequently, a Riemannian metric on the whole $SO(3)$ manifold is induced by the left extension of this inner product. We apply the formula in the Riemannian geometry, $2\langle\langle \nabla_{\boldsymbol{X}} \boldsymbol{Y}, \boldsymbol{Z} \rangle\rangle = \{\langle\langle [\boldsymbol{X}, \boldsymbol{Y}], \boldsymbol{Z} \rangle\rangle - \langle\langle [\boldsymbol{X}, \boldsymbol{Z}], \boldsymbol{Y} \rangle\rangle - \langle\langle [\boldsymbol{Y}, \boldsymbol{Z}], \boldsymbol{X} \rangle\rangle\}$, to the present system, where $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ are left invariant vector fields on $SO(3)$ and $[*, *]$ denotes the Lie bracket [6]. Identifying the Lie algebra $\mathfrak{so}(3)$ with the set of left invariant vector fields $\mathfrak{X}_L(SO(3))$, the covariant derivative at e is derived from the above formula as

$$(\nabla_{\boldsymbol{\eta}} \boldsymbol{\xi})_e = \tfrac{1}{2}\{[\boldsymbol{\eta}, \boldsymbol{\xi}] - \mathbf{I}^{-1}(\mathbf{I}(\boldsymbol{\eta}) \times \boldsymbol{\xi} + \mathbf{I}(\boldsymbol{\xi}) \times \boldsymbol{\eta})\} \tag{2}$$

where $[\boldsymbol{\eta}, \boldsymbol{\xi}] = \boldsymbol{\eta} \times \boldsymbol{\xi}$ for $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathfrak{so}(3)$. Then the Euler's equation (1) is readily obtained as a geodesic equation. To show this, let us denote by $\gamma_{\boldsymbol{u}}(\cdot) : \mathbb{R}^1 \to SO(3)$ the geodesic curve which starts from $g \in SO(3)$ in the direction of $\boldsymbol{u} \in T_g SO(3)$, namely satisfying $\gamma_{\boldsymbol{u}}(0) = g$ and $\dot{\gamma}_{\boldsymbol{u}}(0) := \frac{\mathrm{d}}{\mathrm{d}t}\gamma(0) = \boldsymbol{u}$. Because of the left invariance, the geodesic equation $\nabla_{\dot{\gamma}_u} \dot{\gamma}_{\boldsymbol{u}} = 0$ at $h := \gamma_{\boldsymbol{u}}(t) \in SO(3)$ is equivalent to the following equation at e [10]:

$$T_h L_{h^{-1}}\{\nabla_{\dot{\gamma}_u} \dot{\gamma}_{\boldsymbol{u}}|_h\} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Omega} + (\nabla_{\boldsymbol{\Omega}} \boldsymbol{\Omega})_e = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Omega} + \mathbf{I}^{-1}\{\boldsymbol{\Omega} \times \mathbf{I}(\boldsymbol{\Omega})\} = 0 \tag{3}$$

where $\boldsymbol{\Omega} := T_h L_{h^{-1}} \dot{\gamma}_{\boldsymbol{u}} \in \mathfrak{so}(3)$. It is confirmed that equation (3) is identical to equation (1). Thus, one can investigate the instability of the rigid-body motion by the theory of Riemannian geometry.

Before the instability analysis, we briefly describe the *Jacobi field* and the *conjugate points*, that are significant concepts concerning the geodesic variation. The Jacobi field (or the variational field) $\boldsymbol{V}(t)$ along the geodesic $\gamma_{\boldsymbol{u}}(t)$ is defined by $\boldsymbol{V}(t) := \partial_{\tilde{s}}\alpha(t,\tilde{s})|_{\tilde{s}=0}$, where $\alpha(t,\tilde{s})$ is a variation of $\gamma_{\boldsymbol{u}}$ with a variational parameter $\tilde{s} \in \mathbb{R}^1$ such that $\alpha(t,0) = \gamma_{\boldsymbol{u}}(t)$ and $\nabla_t \partial_t \alpha(t,\tilde{s}) = 0$ for any $\tilde{s}$. This $\boldsymbol{V}(t)$ is governed by the Jacobi equation, $\nabla_{\dot{\gamma}_u(t)}\nabla_{\dot{\gamma}_u(t)}\boldsymbol{V}(t) + R(\boldsymbol{V}(t), \dot{\gamma}_{\boldsymbol{u}}(t))\dot{\gamma}_{\boldsymbol{u}}(t) = 0$, where $R(\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} := \{\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{v}} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{u}} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]}\}\boldsymbol{w}$ is the curvature tensor. Then, it is readily derived that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\!\langle \boldsymbol{V}(t), \boldsymbol{V}(t)\rangle\!\rangle = 2\{\langle\!\langle \nabla_{\dot{\gamma}_u}\boldsymbol{V}(t), \nabla_{\dot{\gamma}_u}\boldsymbol{V}(t)\rangle\!\rangle - K(\boldsymbol{V}(t), \dot{\gamma}_{\boldsymbol{u}}(t))\}$$

where $K(\boldsymbol{u},\boldsymbol{v}) := \langle\!\langle R(\boldsymbol{u},\boldsymbol{v})\boldsymbol{v}, \boldsymbol{u}\rangle\!\rangle$is the *sectional curvature* of the two-dimensional section spanned by the vectors $\boldsymbol{u}, \boldsymbol{v} \in T_g SO(3)$. The norm $\langle\!\langle \boldsymbol{V}, \boldsymbol{V}\rangle\!\rangle$ gives a measure for the instability of the geodesic. For instance, if $K(\boldsymbol{V}(t), \dot{\gamma}_{\boldsymbol{u}}(t))$ is negative definite along $\gamma_{\boldsymbol{u}}$, the geodesic is said to be *unstable* because one of the two independent solutions is growing with respect to the time. If there exists $\boldsymbol{V}(t)$ such that $\boldsymbol{V}(0) = \boldsymbol{V}(t_0) = \boldsymbol{0}$ for $t_0 > 0$, the point $\gamma_{\boldsymbol{u}}(t_0)$ is said to be *conjugate* to the point $\gamma_{\boldsymbol{u}}(0)$ with the conjugate time t_0. Both for the geodesic and its variation, the same element in $SO(3)$ acts on the rigid body at the conjugate time.

3. Instability analysis by the Jacobi equation

It can be easily shown that the Euler equation (1) for $\boldsymbol{\Omega}$ has three steady solutions, $\boldsymbol{S}_1 = (\varpi, 0, 0)^T$, $\boldsymbol{S}_2 = (0, \omega, 0)^T$ and $\boldsymbol{S}_3 = (0, 0, \xi)^T$ where ϖ, ω and ξ are constants. Concerning these steady solutions, the following theorem is well known in the mechanics [2, 4].

If $I_1 < I_2 < I_3$, $\boldsymbol{S}_1$ and $\boldsymbol{S}_3$ are stable while $\boldsymbol{S}_2$ is unstable in the sense of *Liapunov*.

In the following, we investigate the instability of these solutions by the Riemannian geometry introduced above.

First, let us study the development of the Jacobi field. Applying the covariant derivative (2) and equation (3), the general form of the Jacobi equation for the $SO(3)$ manifold is obtained to be

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{Y} + \boldsymbol{X} \times \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Y} - F\left(\boldsymbol{X}, \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Y}\right) + \frac{1}{2}F(\boldsymbol{X},\boldsymbol{X}) \times \boldsymbol{Y} - F(\boldsymbol{X}, \boldsymbol{X}\times\boldsymbol{Y}) = 0 \qquad (4)$$

where $F(\boldsymbol{X},\boldsymbol{Y}) := \mathsf{I}^{-1}\{\mathsf{I}(\boldsymbol{X}) \times \boldsymbol{Y} + \mathsf{I}(\boldsymbol{Y}) \times \boldsymbol{X}\}$, and the two vectors $\boldsymbol{X} = \boldsymbol{X}(t) \in \mathfrak{so}(3)$ and $\boldsymbol{Y} = \boldsymbol{Y}(t) \in \mathfrak{so}(3)$ correspond to the velocity vector and the Jacobi field along the geodesic, respectively. We impose the orthogonal condition $\langle\!\langle \boldsymbol{X}, \boldsymbol{Y}\rangle\!\rangle_e = 0$ because the component of $\boldsymbol{Y}$ parallel to $\boldsymbol{X}$ is only related to the reparametrization of the time t. Then, for a steady state $\boldsymbol{X} = \boldsymbol{S}_2 = (0, \omega, 0)^T$ and general ordering of the magnitudes I_1, I_2 and I_3, the Jacobi field has the form of $\boldsymbol{Y}(t) = (y_1(t), 0, y_3(t))^T$ and equation (4) is explicitly described as

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}y_1 + \frac{I_2 - I_3}{I_1}\omega^2 y_1 + \left(1 - \frac{I_2 - I_3}{I_1}\right)\omega\frac{\mathrm{d}}{\mathrm{d}t}y_3 = 0 \\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}y_3 + \frac{I_2 - I_1}{I_3}\omega^2 y_3 - \left(1 - \frac{I_2 - I_1}{I_3}\right)\omega\frac{\mathrm{d}}{\mathrm{d}t}y_1 = 0.
\end{aligned} \qquad (5)$$

By eliminating y_3 from (5), we obtain the following differential equation for y_1:

$$\frac{\mathrm{d}^4}{\mathrm{d}t^4}y_1 + \left(1 + \frac{(I_2 - I_3)(I_2 - I_1)}{I_1 I_3}\right)\omega^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}y_1 + \frac{(I_2 - I_3)(I_2 - I_1)}{I_1 I_3}\omega^4 y_1 = 0.$$

Assuming $y_1 = A\exp(\lambda t)$ where A and λ are constants, it is readily found that λ^2 has two roots,

$$\lambda_{\mathrm{I}}^2 = -\omega^2 \qquad \text{and} \qquad \lambda_{\mathrm{II}}^2 = -\frac{(I_2 - I_3)(I_2 - I_1)}{I_1 I_3}\omega^2$$

from which the above stability theorem is recovered. In fact, if I_2 is the minimum or maximum of the set $\{I_1, I_2, I_3\}$, only periodic solutions can exist because both λ_j^2's ($j = \mathrm{I}, \mathrm{II}$) are negative. However, in contrast, if $I_1 < I_2 < I_3$ or $I_1 > I_2 > I_3$, there exists y_1 which grows exponentially with respect to the time because λ_{II}^2 becomes positive.

It is remarkable that λ_{I}^2 corresponds to the frequencies of the steady-state rotation observed in the *spatial* frame. This is the unique point of this analysis which treats the group manifold. Although the geodesic equation is identical to the governing equation for the *body* angular velocity, the manifold itself is the *group* $SO(3)$ *which acts on the rigid body*. Therefore, both the frequencies that correspond to the *rotational action* determined by $\boldsymbol{S}_2$ and the *perturbative action* which moves the rotation axis are obtained. In contrast to this, only the latter appears in the conventional stability analysis that simply linearizes equation (3).

4. Conjugate points

It is also worth mentioning that regardless of the shape of the rigid body, a conjugate point exists for the geodesic which corresponds to the linearly stable steady rotation. The general solution of the Jacobi equation (5) is

$$\begin{aligned} y_1 &= A\sin(\omega t) + B\cos(\omega t) + (1-a)b\{C\sin\left(\sqrt{ab}\omega t\right) + D\cos\left(\sqrt{ab}\omega t\right)\} \\ y_3 &= B\sin(\omega t) - A\cos(\omega t) - (1-b)\sqrt{ab}\{D\sin\left(\sqrt{ab}\omega t\right) - C\cos\left(\sqrt{ab}\omega t\right)\} \end{aligned} \tag{6}$$

where A, B, C and D are arbitrary constants, $a = (I_2 - I_3)/I_1$, and $b = (I_2 - I_1)/I_3$. In order to show the existence of a conjugate point, we require $(1-b)\sqrt{ab}C = A$ and $(1-a)bD = -B$ so that $y_1(0) = y_3(0) = 0$. Then, equation (6) is described as

$$\begin{pmatrix} y_1(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} \sin(\omega t) + m\sin\left(\sqrt{ab}\omega t\right) & \cos(\omega t) - \cos\left(\sqrt{ab}\omega t\right) \\ -\left(\cos(\omega t) - \cos\left(\sqrt{ab}\omega t\right)\right) & \sin(\omega t) + m^{-1}\sin\left(\sqrt{ab}\omega t\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{7}$$

where $m = \sqrt{\frac{b}{a}\frac{1-a}{1-b}}$. If the determinant of the 2×2 matrix in equation (7) vanishes at $t = t_0 > 0$, a conjugate point can exist because $y_1(t_0) = y_3(t_0) = 0$ for a certain ratio A/B. This zero-determinant condition is equivalent to

$$\frac{1+M}{2}\cos\left(\left(\sqrt{ab}+1\right)\omega t\right) + \frac{1-M}{2}\cos\left(\left(\sqrt{ab}-1\right)\omega t\right) - 1 = 0 \tag{8}$$

where $M = \frac{1}{2}(m + \frac{1}{m})$. Therefore, if I_2 is the largest or smallest of I_i's ($i = 1, 2, 3$), which is equivalent to $M \geqslant 1$, a conjugate point exists because a solution $t = t_0 (> 0)$ of equation (8) exists.

Furthermore, it is interesting that *any* Jacobi field (7) has a conjugate time, for the rigid bodies of appropriate shapes. In fact, the right-hand side of equation (7) vanishes for any A and B, if and only if

$$\cos(\omega t) - \cos\left(\sqrt{ab}\omega t\right) = \sin(\omega t) + m\sin\left(\sqrt{ab}\omega t\right) = \sin(\omega t) + m^{-1}\sin\left(\sqrt{ab}\omega t\right) = 0. \tag{9}$$

Therefore, a conjugate time $t = t_0 > 0$ which satisfies equation (9) exists, if and only if

$$I_1 = I_3 (\text{which implies } m = 1) \qquad \text{or} \qquad \sqrt{ab} \in \mathbb{Q}. \tag{10}$$

One can easily confirm that there exist many trios (I_1, I_2, I_3)'s which satisfy the second condition in equation (10) together with *a priori* constraints for the general rigid bodies,

$$I_i \geqslant 0 (i = 1, 2, 3) \qquad \text{and} \qquad I_i + I_j \geqslant I_k \qquad (i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i). \tag{11}$$

(For example, $\sqrt{ab} = 1/8$ for $(I_1, I_2, I_3) = (16, 15, 20)$.) If condition (10) is satisfied, both the steadily rotating rigid body (corresponding to the geodesic) and the slightly perturbed one (corresponding to the variation) simultaneously return to their initial configurations, namely are acted by $e \in SO(3)$ at the same time. This statement is consistent with the conventional linear analysis made in the *spatial* coordinate system [4].

5. Sectional curvatures

Secondly, let us study the sectional curvature of the $SO(3)$ manifold. For *infinite*-dimensional Lie–Poisson systems, the sectional curvatures are mainly investigated because of the difficulty of the Jacobi equation [3, 8, 9, 11]. In contrast, for the finite dimension, the Jacobi equation can be given an explicit solution. In the following, we calculate the sectional curvature directly to investigate the relationship between the Riemannian structure and the stability property. In the present formulation, the sectional curvature at the identity element $e \in SO(3)$ is generally described as

$$K(\boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{2}\left(\frac{(I_2 - I_3)^2}{2I_1} + I_2 + I_3 - \frac{3}{2}I_1\right)(x_2 y_3 - x_3 y_2)^2 + (\textit{cyclic permutations})$$

where $\boldsymbol{X}, \boldsymbol{Y} \in \mathfrak{so}(3)$. Then, for the steady state $\boldsymbol{S}_2$,

$$K(\boldsymbol{S}_2, \boldsymbol{Y}) = \frac{\omega^2}{2}\left\{\left(\frac{(I_2 - I_3)^2}{2I_1} + I_2 + I_3 - \frac{3}{2}I_1\right)y_3^2 + \left(\frac{(I_1 - I_2)^2}{2I_3} + I_1 + I_2 - \frac{3}{2}I_3\right)y_1^2\right\} \tag{12}$$

where $\boldsymbol{Y} = (y_1, y_2, y_3)^T$ corresponds to the Jacobi field. The sign of equation (12) is determined by the coefficients of y_3^2 and y_1^2. If both of them are definitely positive, K is positive for *any* Jacobi field. In contrast, if one of them is negative, the positivity of K is lost. To illustrate the sign of K clearly, let us regard the set (I_1, I_2, I_3) as coordinates of $\mathbb{R}^3$, and denote the surfaces $(I_2-I_3)^2+2I_1I_2+2I_3I_1-3I_1^2 = 0$ and $(I_1-I_2)^2+2I_3I_1+2I_2I_3-3I_3^2 = 0$ by $sf3$ and $sf1$, respectively. These two surfaces together with the boundaries determined by equation (11) are illustrated in figure 1. The classical stability theorem states that $\boldsymbol{S}_2$ is stable if the *point* $\mathbf{I} = (I_1, I_2, I_3)$ is in the regions $S_\alpha i$'s $(\alpha = p, *; i = 0, 1, 2)$, while unstable in $U_\alpha i$'s $(\alpha = p, *; i = 1, 2)$. On the other hand, the sign of K *is positive for any* $\boldsymbol{Y}$ in the regions with the subscript p, while the positivity of K is not assured in those with the subscript $*$. We are interested in the regions $S_* i$'s $(i = 1, 2)$ because the positivity of K is not assured while the geodesic is *stable* by the theorem.

To investigate this situation, let us consider planar rigid bodies in S_*2 such that $I_2 + I_3 = I_1 (0 < I_2 < I_3)$ and the Jacobi field $\boldsymbol{Y}$ with the condition $\boldsymbol{Y}|_{t=0} = \mathbf{0}$. Then, the sectional curvature is explicitly written as

$$K(\boldsymbol{S}_2, \boldsymbol{Y}) = \omega^2\left(-\frac{I_2 I_3}{I_2 + I_3}y_3^2 + I_2 y_1^2\right) \tag{13}$$

where $y_1 = 2\sqrt{-a}C\sin(\omega t) + (1 - a)\{D\cos(\omega t) - C\sin(\sqrt{-a}\omega t) - D\cos(\sqrt{-a}\omega t)\}$, $y_3 = (1 - a)D\sin(\omega t) + 2\sqrt{-a}\{-C\cos(\omega t) - D\sin(\sqrt{-a}\omega t) + C\cos(\sqrt{-a}\omega t)\}$ and $a = (I_2-I_3)/(I_2+I_3)$. Varying the values of the parameters I_2, I_3, C and D, we investigated

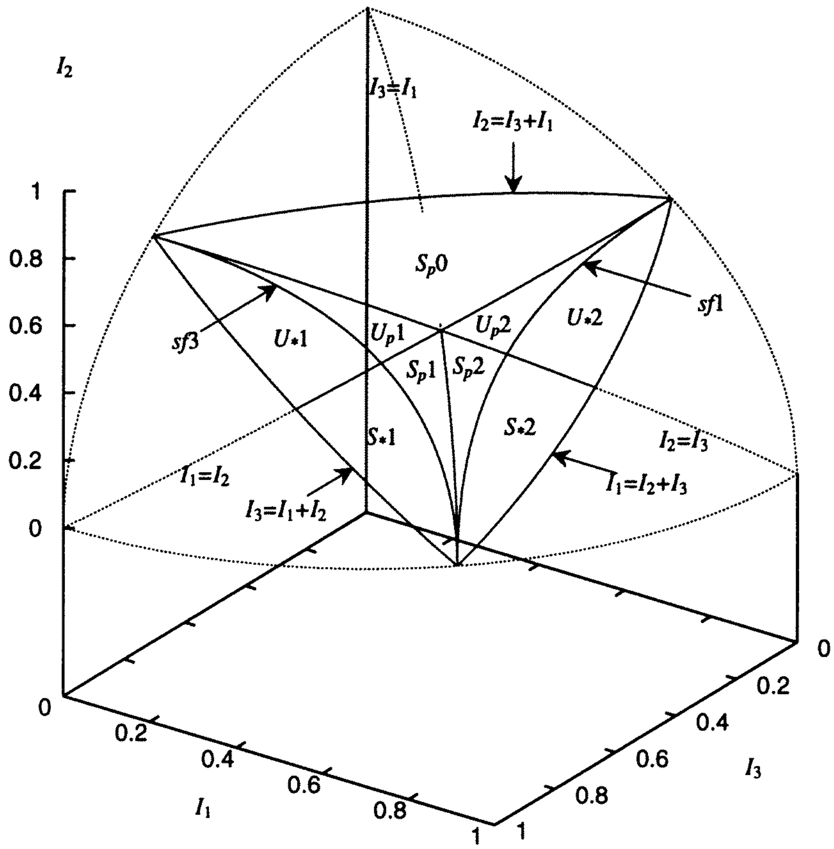

Figure 1. Intersections of the unit sphere in $\mathbb{R}^3$ with surfaces concerning the instability of $\boldsymbol{S}_2$. The sectional curvature K is *positive definite* in the regions with the subscript p, while the positivity of K is lost in those with the subscript $*$.

the temporal behaviour of equation (13). Then, it is confirmed that the sectional curvature K has positive maxima during its time development. For example, if $I_2 \cong I_3$ (the rigid body takes the form of a nearly circular plate), $K(\boldsymbol{S}_2, \boldsymbol{Y})$ takes positive maxima with larger amplitude and negative minima with smaller amplitude (figure 2). In contrast, if $I_2 \ll I_3$ (the rigid body being like a stick), $K(\boldsymbol{S}_2, \boldsymbol{Y})$ takes negative and positive values quasiperiodically. We further confirmed that the maxima of $K(\boldsymbol{S}_2, \boldsymbol{Y})$ are positive at any point in S_*i's ($i = 1, 2$).

It is interesting to find that not only the maxima but also the time average $\bar{K}$ of $K(\boldsymbol{S}_2, \boldsymbol{Y})$ is positive in $S_\alpha i$'s ($\alpha = p, *; i = 0, 1, 2$). This statement is trivial in $S_p i$'s ($i = 0, 1, 2$). As for the regions S_*i's ($i = 1, 2$), a straightforward calculation shows that $K(\boldsymbol{S}_2, \boldsymbol{Y}) = K_c + F(t)$ where $F(t)$ is a function of t with Fourier frequencies 2ω, $(1 \pm \sqrt{ab})\omega$ and $2\sqrt{ab}\omega$, while K_c is a constant given by

$$K_c = \omega^2 \mathrm{M}\{F_1 C^2 + F_2 D^2\}$$

where C and D are the constants which appeared in equation (6),

$$F_1 = \left\{ \frac{(I_2 - I_3)^2}{2I_1} + I_2 + I_3 - \frac{3}{2} I_1 \right\} \frac{I_1}{I_3}(I_2 - I_3) + \left\{ \frac{(I_1 - I_2)^2}{2I_3} + I_1 + I_2 - \frac{3}{2} I_3 \right\}$$

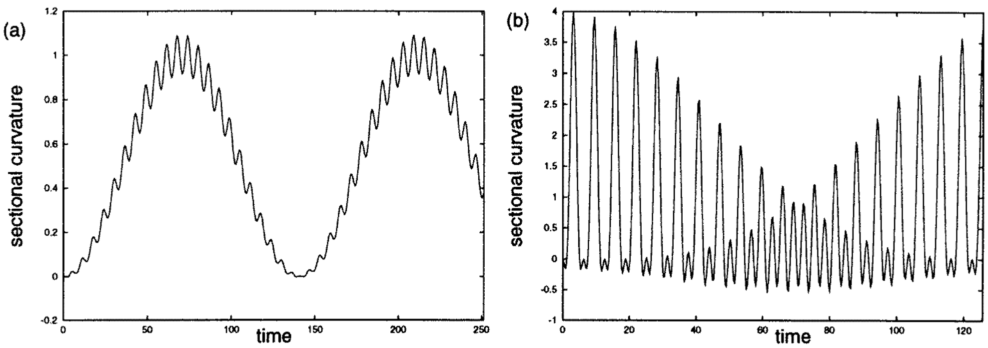

Figure 2. The time evolution of the sectional curvature of an almost circular plate. (a) $\omega = 1$, $I_2 = 1$, $I_3 = 1.001$, $C = 1$ and $D = 0$; (b) $\omega = 1$, $I_2 = 1$, $I_3 = 1.001$, $C = 0$ and $D = 1$.

$$\times \frac{I_2 I_3 - 2I_3 I_1 + I_1 I_2}{2I_3}$$

$$F_2 = \left\{ \frac{(I_1 - I_2)^2}{2I_3} + I_1 + I_2 - \frac{3}{2} I_3 \right\} (I_2 - I_1) + \left\{ \frac{(I_2 - I_3)^2}{2I_1} + I_2 + I_3 - \frac{3}{2} I_1 \right\}$$
$$\times \frac{I_2 I_3 - 2I_3 I_1 + I_1 I_2}{2I_3}$$

and $M = \frac{1}{2}(I_2 - I_1)(I_3 + I_1 - I_2)^2 I_3^{-2} I_1^{-2}$. To show the positiveness of $\bar{K}$, first, let us consider such a rigid body that satisfies the condition $\sqrt{ab} = m/n$ $(m, n \in \mathbb{N})$. Then, taking the time average for the period $2n\pi\omega^{-1}$, the mean curvature $\bar{K}$ is obtained to be $\bar{K} = K_c + \frac{\omega}{2n\pi} \int_0^{2n\pi/\omega} F(t)\,\mathrm{d}t = K_c$. The sign of K_c is positive. In fact, if we denote the region $\bigcup_{i=1,2} S_* i$ by $\boldsymbol{B}$, both MF_1 and MF_2 attain their maxima and minima at the boundary $\partial\boldsymbol{B}$ because $\nabla_{\mathbf{I}}(MF_1)$ and $\nabla_{\mathbf{I}}(MF_2)$ do not vanish in $\boldsymbol{B} - \partial\boldsymbol{B}$. After straightforward calculations, one can confirm that MF_i's $(i = 1, 2)$ are non-negative at $\partial\boldsymbol{B}$, especially positive definite at sf_i's $(i = 3, 1)$. Therefore, $\bar{K}(= K_C)$ is positive in this case. Next, let us examine the rigid body which violates the condition $\sqrt{ab} = m/n$ $(m, n \in \mathbb{N})$; in this case, $\sqrt{ab}$ becomes an irrational number. In order to calculate $\bar{K}$, let us adopt an approximation such that $\sqrt{ab} \approx m/n$ $(m, n \in \mathbb{N})$. (It is known that for any given accuracy, an irrational number can be approximated by a rational number.) For sufficiently large n, which means high enough precision, $\bar{K}$ becomes positive because the integral $\frac{\omega}{2n\pi} \int_0^{2n\pi/\omega} F(t)\,\mathrm{d}t$ becomes small enough.

In the linearly unstable regions $U_\alpha i$'s $(\alpha = p, *; i = 1, 2)$, the above statement about the positiveness of $\bar{K}$ is not valid because $\bar{K}$ can diverge to negative infinity for such $\boldsymbol{Y}(t)$ that grows exponentially with respect to the time, according to equation (13).

6. Conclusion

Geometrical aspects of the free rotation of a rigid body are studied in view of the integrable property known in the classical mechanics. According to the geodesic formulation of the Euler equation, the covariant derivative and the sectional curvature K are given explicitly. By deriving the solution of the Jacobi field $\boldsymbol{Y}$, the stability theorem is recovered and the existence of the conjugate points is confirmed. Furthermore, it is found that the sectional curvature K is dominated by *positive* values in the case of linear stability, though there exist $\boldsymbol{Y}$'s which make K negative in the parts of linearly stable regions $S_* i$'s $(i = 1, 2)$. However, the time-averaged curvature $\bar{K}$ is surely positive in $S_\alpha i$'s ($\alpha = p, *; i = 0, 1, 2$). In contrast, in the linearly unstable regions $U_\alpha i$'s ($\alpha = p, *; i = 1, 2$), there exist always $\boldsymbol{Y}$'s which make $\bar{K}$ grow indefinitely with negative sign. Thus, it is shown that the positivity of $\bar{K}$ for any $\boldsymbol{Y}$ corresponds to the linear stability of the free-rigid-body motion. The dynamical property (stability) is directly explained by Riemannian geometrical quantities ($\boldsymbol{Y}$ and K) in this work.

Since the conjugate points have a close relation to the global nature of the manifold, they will also give clues to investigate the long-time behaviour or the large variations of the geodesics. Further analysis concerning the nonlinear stability should be developed in the near future.

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